B.Sc Part I (Honors)

Roots and co-efficients.

Example.

Q → Find the condition that the roots of the equation $ax^3+3bx^2+3cx+d=0$ are in (a) A.P (b) G.P. (c) H.P.

Ans: → Let $\alpha-\beta, \alpha, \alpha+\beta$ (in A.P.) be three roots of the given equation

$$\therefore \text{Sum of roots} = \alpha-\cancel{\beta}+\alpha+\alpha+\cancel{\beta} = -\frac{3b}{a}$$

$$\Rightarrow 3\alpha = -\frac{3b}{a} \Rightarrow \alpha = -b/a.$$

Since α is a root of the given equation.

$$\therefore a\alpha^3+3b\alpha^2+3c\alpha+d=0$$

Putting $\alpha=-b/a$, we get

$$a\left(-\frac{b}{a}\right)^3+3b\left(-\frac{b}{a}\right)^2+3c\left(-\frac{b}{a}\right)+d=0$$

$$\Rightarrow -a\frac{b^3}{a^3}+3b\frac{b^2}{a^2}-\frac{3bc}{a}+d=0$$

$$\Rightarrow -\frac{b^3}{a^2}+\frac{3b^3}{a^2}-\frac{3bc}{a}+d=0$$

$$\Rightarrow \frac{2b^3}{a^2}-\frac{3bc}{a}+d=0 \Rightarrow 2b^3-3abc+a^2d=0$$

(ii) Let α, β, γ are the roots of the given equation. Where α, β, γ are in G.P

Let $\alpha = \frac{p}{r}$, $\beta = p$ and $\gamma = pr$

Product of roots $\alpha\beta\gamma = -d/a$

$$\Rightarrow \frac{p}{r}.p.pr = -d/a$$

$$\Rightarrow p^3 = -d/a. \quad \text{——— (i)}$$

$\because p=\beta$ is a root of the given equation so we have

$$ap^3+3bp^2+3cp+d=0$$

$$\Rightarrow a\left(-\frac{d}{a}\right)+3bp^2+3cp+d=0 \quad \left[p^3=-\frac{d}{a}\right]$$

$$\Rightarrow -d+3bp^2+3cp+d=0 \Rightarrow 3bp^2+3cp=0$$

$$\Rightarrow 3p(bp+c)=0 \qquad [\because 3p\neq 0]$$

$$\therefore bp=-c \qquad \Rightarrow b^3p^3=-c^3$$

$$b^3\left(-\frac{d}{a}\right)=-c^3 \qquad \left[\because p^3=-d/a\right]$$

$$\Rightarrow b^3d-c^3a=0$$

(ii) Let α, β, γ (in H.P) are three roots of given equation

$$\left.\begin{aligned} \therefore \alpha+\beta+\gamma &= -3b/a \\ \alpha\beta+\beta\gamma+\gamma\alpha &= 3c/a \\ \text{and } \alpha\beta\gamma &= -d/a \end{aligned}\right] \text{———} (1)$$

When α, β, γ are in H.P then

$$\frac{1}{\alpha}+\frac{1}{\gamma}=\frac{2}{\beta}$$

$$\Rightarrow \alpha\beta+\beta\gamma=2\gamma\alpha \Rightarrow \alpha\beta+\beta\gamma+\gamma\alpha=3\gamma\alpha$$

$$\Rightarrow \frac{3c}{a}=3\gamma\alpha \text{ (using 1)} \quad \Rightarrow \gamma\alpha=c/a$$

By (1) $\alpha\beta\gamma=-d/a \Rightarrow \frac{c}{a}\beta=-d/a$

$$\Rightarrow \beta=-d/c$$

As β is the root of the given equation then $a\beta^3+3b\beta^2+3c\beta+d=0$

$$a\left(-\frac{d}{c}\right)^3+3b\left(\frac{d}{c}\right)^2+3c\left(-\frac{d}{c}\right)+d=0$$

$$\Rightarrow -a\frac{d^3}{c^3}+3b\frac{d^2}{c^2}-3d+d=0$$

$$\Rightarrow -a\frac{d^3}{c^3}+\frac{3bd^2}{c^2}-2d=0$$

$$\Rightarrow -d\left(ad^2-3bcd+2c^3\right)=0$$

$$\Rightarrow ad^2-3bcd+2c^3=0 \quad [\because d\neq 0].$$